Approximation / Interpolation Notes

E.g. Compute the quadratic polynomial interpolating $\{(0,3), (1,7), (2,37)\}$ using the Method of Undetermined Coefficients, the Lagrange Basis and the Newton Basis.

Soln:

Method of Undetermined Coefficients:

$$P(x) = \sum_{i=0}^{n} a_i x^i \leftarrow \text{Monomial Basis}$$

In this case, because we have 3 points, $n=2$.

$$P(x) = \sum_{i=0}^{2} a_i x^i$$

To find the a_i's, we'll use the Vandermonde Matrix.

$$\begin{bmatrix} (x_0)^0 & (x_0)^1 & (x_0)^2 \\ (x_1)^0 & (x_1)^1 & (x_1)^2 \\ (x_2)^0 & (x_2)^1 & (x_2)^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_{\text{Vandermonde Matrix}} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 37 \end{bmatrix}$$

Let V be the Vandermonde Matrix.
We can use $PV = LU$ to solve for $\bar{a}$.

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$L_1 V = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 4 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_1 L_1 V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$L_2 P_1 L_1 V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} \leftarrow u$$

$L_2 P_2 L_1 V$ is equivalent to $L_2 \underbrace{P_2 L_1 P_1}_{\hat{L}_1} P_1 V$

$$P_1 L_1 P_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \leftarrow \hat{L}_1$$

$$\begin{aligned} L &= \hat{L}_1^{-1} \cdot L_2^{-1} \\ &= \hat{L}_1^{-1} + L_2^{-1} - I \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix} \end{aligned}$$

$P = P_1$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Now, we have P, L and U.

$V\bar{a} = \bar{b} \Leftrightarrow PV\bar{a} = P\bar{b}$

$\Leftrightarrow LU\bar{a} = P\bar{b}$

Let $U\bar{a} = \bar{d}$

$LU\bar{a} = P\bar{b} \Leftrightarrow L\bar{d} = P\bar{b}$

In $L\bar{d} = P\bar{b}$, we solve for $\bar{d}$.
In $U\bar{a} = \bar{d}$, we solve for $\bar{a}$.

$$P\bar{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 37 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 37 \\ 7 \end{bmatrix}$$

$L\bar{d} = P\bar{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 37 \\ 7 \end{bmatrix}$$

$$\left.\begin{array}{l} d_1 = 3 \\ d_1 + d_2 = 37 \rightarrow d_2 = 34 \\ d_1 + \frac{d_2}{2} + d_3 = 7 \rightarrow d_3 = -13 \end{array}\right\} \bar{d} = \begin{bmatrix} 3 \\ 34 \\ -13 \end{bmatrix}$$

$U\bar{a} = \bar{d}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 34 \\ -13 \end{bmatrix}$$

$a_0 = 3$

$a_2 = 13$

$2a_1 + 4a_2 = 34$

$2a_1 + 52 = 34$

$2a_1 = -18$

$a_1 = -9$

$$\bar{a} = \begin{bmatrix} 3 \\ -9 \\ 13 \end{bmatrix}$$

$$P(x) = \sum_{i=0}^{2} a_i x^i$$
$$= 3 - 9x + 13x^2$$

We can check:

$P(0) = 3 - 9(0) + 13(0^2)$
$= 3$
$= y(0)$

$P(1) = 3 - 9(1) + 13(1^2)$
$= 7$
$= y(1)$

$P(2) = 3 - 9(2) + 13(2^2)$
$= 37$
$= y(2)$

Lagrange Basis:

$$P(x) = \sum_{i=0}^{n} l_i(x)\, y_i$$

where $$l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}$$

$$l_0(x) = \frac{(x-1)}{(0-1)} \cdot \frac{(x-2)}{(0-2)}$$

$$= \frac{(x-1)(x-2)}{2}$$

$$l_1(x) = \frac{(x-0)}{(1-0)} \cdot \frac{(x-2)}{(1-2)}$$

$$= -x(x-2)$$

$$l_2(x) = \frac{(x-0)}{(2-0)} \cdot \frac{(x-1)}{(2-1)}$$

$$= \frac{x(x-1)}{2}$$

$$P(x) = \frac{3(x-1)(x-2)}{2} - 7x(x-2) + \frac{37x(x-1)}{2}$$

$$P(0) = \frac{3(-1)(-2)}{2}$$
$$= 3$$
$$= y(0)$$

$$P(1) = -7(-1)$$
$$= 7$$
$$= y(1)$$

$$P(2) = \frac{37(2)(2-1)}{2}$$
$$= 37$$
$$= y(2)$$

Note: Let's expand and simplify $P(x)$.

$$P(x) = \frac{3(x-1)(x-2)}{2} - 7x(x-2) + \frac{37x(x-1)}{2}$$
$$= \frac{3}{2}(x^2-3x+2) - 7(x^2-2x) + \frac{37}{2}(x^2-x)$$
$$= \frac{3x^2}{2} - \frac{9x}{2} + 3 - 7x^2 + 14x + \frac{37x^2}{2} - \frac{37x}{2}$$
$$= 3 - 9x + 13x^2$$ ← Same as the monomial basis.

Newton Basis:

$$P(x) = \sum_{i=0}^{n} \left[a_i \prod_{j=0}^{i-1} (x - x_j) \right]$$

where a_i is the i^{th} divided difference on $[x_0, x_1, \ldots, x_i]$

$$P(x) = a_0 + (x - x_0) a_1 + (x - x_0)(x - x_1) a_2 + \ldots$$

where $a_0 = y[x_0] = y_0$

$a_1 = y[x_1, x_0] = \dfrac{y_1 - y_0}{x_1 - x_0}$

x	$y[x_0]$	$y[x_1, x_0]$	$y[x_2, x_1, x_0]$
0	3		
		$\frac{7-3}{1-0} = 4$	
			$\frac{30-4}{2-0} = 13$
1	7		
		$\frac{37-7}{2-1} = 30$	
2	37		

$a_0 = 3, \quad a_1 = 4, \quad a_2 = 13$

$P(x) = 3 + 4x + 13x(x-1)$

$P(0) = 3$
$= y(0)$

$P(1) = 3+4$
$= 7$
$= y(1)$

$P(2) = 3+4(2) + 13(2)(1)$
$= 3+8+26$
$= 37$
$= y(2)$

Let's expand $P(x)$.
$P(x) = 3 + 4x + 13x(x-1)$
$= 3 + 4x - 13x + 13x^2$
$= 3 - 9x + 13x^2$